C Notes

In the following Appendices we provide more detailed information on the simulations used to calculate the quantities discussed in Secs. 3–10. We present this information only for results that are new w.r.t. FLAG 19. For all other results the information is available in the corresponding Appendices B.1–8 in FLAG 19 [1] and B.1–7 of FLAG 16 [2]. The complete information is available on the FLAG website http://flag.unibe.ch [3].

C.1 Notes to Sec. 3 on quark masses

Collab.	Ref.	N_f	a [fm]	Description
ETM 21A	[4]	2+1+1	0.07, 0.08, 0.09	Scale set from w_0.

Table 79: Continuum extrapolations/estimation of lattice artifacts in determinations of m_{ud}, m_s and, in some cases m_u and m_d, with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	a [fm]	Description
ALPHA 19	[5]	2+1	0.05, 0.064, 0.076, 0.086	

Table 80: Continuum extrapolations/estimation of lattice artifacts in determinations of m_{ud}, m_s and, in some cases m_u and m_d, with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
ETM 21A	[4]	2+1+1	physical	

Table 81: Chiral extrapolation/minimum pion mass in determinations of m_{ud}, m_s and, in some cases, m_u and m_d, with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
ALPHA 19	[5]	2+1	198	

Table 82: Chiral extrapolation/minimum pion mass in determinations of m_{ud}, m_s and, in some cases m_u and m_d, with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
ETM 21A	[4]	2+1+1	2.5 – 5.6	3.6	

Table 83: Finite-volume effects in determinations of m_{ud}, m_s and, in some cases m_u and m_d, with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
ALPHA 19	[5]	2+1	2.4 – 4.1	$\gtrsim 4.0$	

Table 84: Finite-volume effects in determinations of m_{ud}, m_s and, in some cases m_u and m_d, with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	Description
ETM 21A	[4]	2+1+1	Nonperturbative (RI/MOM)

Table 85: Renormalization in determinations of m_{ud}, m_s and, in some cases m_u and m_d, with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	Description
ALPHA 19	[5]	2+1	Schrödinger functional.

Table 86: Renormalization in determinations of m_{ud}, m_s and, in some cases m_u and m_d, with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	a [fm]	Description
ETM 21A	[4]	2+1+1	0.07, 0.08, 0.09	Scale set from w_0. Twisted mass action for charm quarks.
HPQCD 20A	[6]	2+1+1	0.03, 0.042,0.06, 0.09, 0.12	Scale set from w_0 and f_π. HISQ action for charm quarks.

Table 87: Continuum extrapolations/estimation of lattice artifacts in the determinations of m_c with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	a [fm]	Description
ALPHA 21	[7]	2+1	0.039 to 0.087 fm	Wilson-clover action for the charm quark.
Petreczky 19	[8]	2+1	0.025 to 0.11 fm	HISQ action for the charm quark. Scale set from r_1 parameter of heavy quark potential and f_π.

Table 88: Continuum extrapolations/estimation of lattice artifacts in the determinations of m_c with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
ETM 21A	[4]	2+1+1	physical	
HPQCD 20A	[6]	2+1+1	physical	

Table 89: Chiral extrapolation/minimum pion mass in the determinations of m_c with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
ALPHA 21	[7]	2+1	198	
Petreczky 19	[8]	2+1	161	

Table 90: Chiral extrapolation/minimum pion mass in the determinations of m_c with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$	Description
ETM 21A	[4]	2+1+1	2.5 – 5.6	3.6	
HPQCD 20A	[6]	2+1+1	1.9 – 5.76	3.8	

Table 91: Finite-volume effects in the determinations of m_c with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$	Description
ALPHA 21	[7]	2+1	2.5 – 4.0	4.2	
Petreczky 19	[8]	2+1	1.6 – 5.2	4.2	

Table 92: Finite-volume effects in the determinations of m_c with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	Description
ETM 21A	[4]	2+1+1	Nonperturbative (RI/MOM)
HPQCD 20A	[6]	2+1+1	Nonperturbative (RI/SMOM)

Table 93: Renormalization in the determinations of m_c with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	Description
ALPHA 21	[7]	2+1	Schrödinger functional
Petreczky 19	[8]	2+1	not required

Table 94: Renormalization in the determinations of m_c with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	a [fm]	Description
HPQCD 21	[9]	2+1+1	0.03, 0.042, 0.06, 0.09	Scale set from w_0 and f_π. HISQ action for charm quarks.

Table 95: Continuum extrapolations/estimation of lattice artifacts in the determinations of m_b with $N_f = 2 + 1 + 1$ quark flavours.

Collab.	Ref.	N_f	a [fm]	Description
Petreczky 19	[8]	2+1	0.025 to 0.11 fm	HISQ action for the charm quark. Scale set from r_1 parameter of heavy quark potential and f_π.

Table 96: Continuum extrapolations/estimation of lattice artifacts in the determinations of m_b with $N_f = 2 + 1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]
HPQCD 21	[9]	2+1+1	physical

Table 97: Chiral extrapolation/minimum pion mass in the determinations of m_b with $N_f = 2 + 1 + 1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]
Petreczky 19	[8]	2+1	161

Table 98: Chiral extrapolation/minimum pion mass in the determinations of m_b with $N_f = 2 + 1$ quark flavours.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$
HPQCD 21	[9]	2+1+1	1.9 - 5.76	3.8

Table 99: Finite-volume effects in the determinations of m_b with N_f = 2+1+1 quark flavours.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$
Petreczky 19	[8]	2+1	1.6 – 5.2	4.2

Table 100: Finite-volume effects in the determinations of m_b with $N_f = 2 + 1$ quark flavours.

Collab.	Ref.	N_f	Description
ETM 21A	[4]	2+1+1	Nonperturbative (RI/MOM)

Table 101: Lattice renormalization in the determinations of m_b with $N_f = 2 + 1 + 1$ flavours.

Collab.	Ref.	N_f	Description
Petreczky 19	[8]	2+1	not required

Table 102: Lattice renormalization in the determinations of m_b with $N_f = 2 + 1$ flavours.

C.2 Notes to Sec. 4 on $|V_{ud}|$ and $|V_{us}|$

Collab.	Ref.	N_f	a [fm]	Description
FNAL/MILC 18	[10]	2+1+1	0.042, 0.06, 0.09, 0.12, 0.15	HISQ quark action. Relative scale through r_1.
PACS 19	[11]	2+1	0.085	Nonperturbative $\mathcal{O}(a)$ clover quark action. Scale set from Ξ-baryon mass.
PACS 22	[12]	2+1	0.085, 0.063	Nonperturbative $\mathcal{O}(a)$ clover quark action. Scale set from Ξ-baryon mass.

Table 103: Continuum extrapolations/estimation of lattice artifacts in the determinations of $f_+(0)$.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
FNAL/MILC 18	[10]	2+1+1	$144_{\rm RMS}(135_{\pi,5})$	Chiral interpolation through NLO $SU(3)$ PQ staggered χPT with continuum χPT at NNLO. Lightest Nambu-Goldstone mass is 135 MeV and lightest RMS mass is 144 MeV at the same gauge ensemble with $a \simeq 0.06$ fm.
PACS 19	[11]	2+1	135	Physical point simulation at a single pion mass 135 MeV.
PACS 22	[12]	2+1	135	Physical point simulation at a single pion mass 135 MeV.

Table 104: Chiral extrapolation/minimum pion mass in determinations of $f_+(0)$. The subscripts RMS and $\pi, 5$ in the case of staggered fermions indicate the root-mean-square mass and the Nambu-Goldstone boson mass, respectively.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$	Description
FNAL/MILC 18	[10]	2+1+1	2.6–5.8	$4.2_{\rm RMS}(3.9_{\pi,5})$	The values correspond to $M_{\pi,\rm RMS}$ = 144 MeV and $M_{\pi,5} = 135$ MeV, respectively.
PACS 19	[11]	2+1	10.9	7.5	
PACS 22	[12]	2+1	10.9	7.5	

Table 105: Finite-volume effects in determinations of $f_+(0)$. The subscripts RMS and $\pi, 5$ in the case of staggered fermions indicate the root-mean-square mass and the Nambu-Goldstone boson mass, respectively.

Collab.	Ref.	N_f	a [fm]	Description
ETM 21	[13]	2+1+1	0.07, 0.08, 0.09	Wilson-clover twisted mass quark action. Relative scale through gradient flow scale w_0 and absolute scale through f_π.
CalLat 20	[14]	2+1+1	0.06, 0.09, 0.12, 0.15	Möbius domain-wall valence quarks on gradient-flowed HISQ ensembles. Relative scale through the gradient flow scale w_0.

Table 106: Continuum extrapolations/estimation of lattice artifacts in determinations of f_K/f_π for $N_f = 2+1+1$ simulations.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
ETM 21	[13]	2+1+1	134	Chiral extrapolation based on NLO $SU(2)$ χPT.
CalLat 20	[14]	2+1+1	157	Chiral extrapolation based on NNLO $SU(3)$ χPT. We quote the root-mean-square (RMS) mass of the valence and valence-sea pions as $M_{\pi,\rm min}$. The smallest mass is 176 MeV for the HISQ sea pions, which do not enter until NNLO in the χPT expression.

Table 107: Chiral extrapolation/minimum pion mass in determinations of f_K/f_π for $N_f = 2+1+1$ simulations.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$	Description
ETM 21	[13]	2+1+1	2.0–5.6	3.8	Three different volumes at $M_\pi =$ 253 MeV and $a = 0.08$ fm.
CalLat 20	[14]	2+1+1	2.4–7.2	3.8	Three different volumes at $M_\pi =$ 220 MeV and $a = 0.12$ fm.

Table 108: Finite-volume effects in determinations of f_K/f_π for $N_f = 2+1+1$.

C.3 Notes to section 5 on low-energy constants

Collab.	Ref.	N_f	a [fm]	Description
ETM 21A	[4]	2+1+1	0.095, 0.082, 0.069	Scale set by $f_\pi = 130.4(2)$ MeV.
ETM 21	[13]	2+1+1	0.092, 0.080, 0.068	Scale set by $f_\pi = 130.4(2)$ MeV.
χQCD 21	[15]	2+1	0.063, 0.071, 0.084, 0.114	Same configs. as RBC/UKQCD 15E.
Wang 16	[16]	2+1	0.113	Same configs. as RBC/UKQCD 08A.
ETM 20A	[17]	2	0.0914(15)	Single lattice spacing.

Table 109: Continuum extrapolations/estimation of lattice artifacts in determinations of the $SU(2)$ low-energy constants $\Sigma, F, \bar{\ell}_4, \bar{\ell}_6$, and $SU(3)$ low-energy constants Σ_0, F_0.

Collab.	Ref.	N_f	a [fm] or a^{-1} [GeV]	Description
Gao 21	[18]	2+1	0.04, 0.06, 0.076	One lattice spacing at phys. pt.
χQCD 20	[19]	2+1	0.083–0.195	One lattice spacing below 0.1 fm.
Feng 19	[20]	2+1	1.015, 1.378, 1.730	$a > 0.1$ fm.

Table 110: Continuum extrapolations/estimation of lattice artifacts in determinations of the low-energy constants related to the vector form factor of the pion.

Collab.	Ref.	N_f	a [fm]	Description
ETM 20B	[21]	2	0.0914(15)	Single lattice spacing.
Mai 19	[22]	2	0.12	Single lattice spacing.
Culver 19	[23]	2	0.12	Single lattice spacing.

Table 111: Continuum extrapolations/estimation of lattice artifacts in determinations of the low-energy constants related to $\pi\pi$ scattering.

Collab.	Ref.	N_f	a [fm]	Description
ETM 18B	[24]	2+1+1	0.089, 0.082, 0.062	Same configuration with ETM 17G.
ETM 17G	[25]	2+1+1	0.089, 0.082, 0.062	Scale set by the Sommer parameter r_0.
PACS-CS 13	[26]	2+1	0.09	Single lattice spacing.
Fu 11A	[27]	2+1	0.15	Single lattice spacing.
NPLQCD 07B	[28]	2+1	0.09, 0.125	Configurations generated by MILC.
NPLQCD 06B	[29]	2+1	0.125	Single lattice spacing.

Table 112: Continuum extrapolations/estimation of lattice artifacts in determinations of the low-energy constants related to πK scattering.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
ETM 21A	[4]	2+1+1	134	4 pion masses in $[134, 346]$ MeV.
ETM 21	[13]	2+1+1	135	4 pion masses in $[134, 346]$ MeV.
χQCD 21	[15]	2+1	139	3 pion masses with different a.
Wang 16	[16]	2+1	220	8 (3) pion masses in val (sea) sector.
ETM 20A	[17]	2	132	

Table 113: Chiral extrapolation/minimum pion mass in determinations of the $SU(2)$ low-energy constants $\Sigma, F, \bar{\ell}_4, \bar{\ell}_6$, and $SU(3)$ low-energy constants Σ_0, F_0.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
Gao 21	[18]	2+1	140	
χQCD 20	[19]	2+1	139	
Feng 19	[20]	2+1	~ 135	

Table 114: Chiral extrapolation/minimum pion mass in determinations of the low-energy constants related to the vector form factor of the pion.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
ETM 20B	[21]	2	134	2 pion masses.
Mai 19	[22]	2	224	
Culver 19	[23]	2	226	2 pion masses.

Table 115: Chiral extrapolation/minimum pion mass in determinations of the low-energy constants related to $\pi\pi$ scattering.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
ETM 18B	[24]	2+1+1	276	5 pion masses in $[230, 450]$ MeV.
ETM 17G	[25]	2+1+1	276	5 pion masses in $[230, 450]$ MeV.
PACS-CS 13	[26]	2+1	166	5 pion masses in $[166, 707]$ MeV.
Fu 11A	[27]	2+1	590 (RMS)	6 valence pion masses.
NPLQCD 07B	[28]	2+1	413 (RMS)	4 pion masses.
NPLQCD 06B	[29]	2+1	488 (RMS)	4 pion masses.

Table 116: Chiral extrapolation/minimum pion mass in determinations of the low-energy constants related to πK scattering.

Collab.	Ref.	N_f	L [fm]	#V
ETM 21A	[4]	2+1+1	5.52	2
ETM 21	[13]	2+1+1	5.55	2
χQCD 21	[15]	2+1	5.4	2 at physical point.
Wang 16	[16]	2+1	2.7	1
ETM 20A	[17]	2	4.39	2

Table 117: Finite-volume effects in determinations of the $SU(2)$ low-energy constants $\Sigma, F, \bar\ell_4, \bar\ell_6$, and $SU(3)$ low-energy constants Σ_0, F_0.

Collab.	Ref.	N_f	L [fm]	#V
Gao 21	[18]	2+1	4.86	1
χQCD 20	[19]	2+1	6.24	3
Feng 19	[20]	2+1	6.22	3

Table 118: Finite-volume effects in determinations of the low-energy constants related to the vector form factor of the pion.

Collab.	Ref.	N_f	L [fm]	#V
ETM 20B	[21]	2	2.92	2
Mai 19	[22]	2	2.88	1
Culver 19	[23]	2	2.88	3

Table 119: Finite-volume effects in determinations of the low-energy constants related to $\pi\pi$ scattering.

Collab.	Ref.	N_f	L [fm]	#V
ETM 18B	[24]	2+1+1	2.832	2
ETM 17G	[25]	2+1+1	2.832	2
PACS-CS 13	[26]	2+1	2.9	1
Fu 11A	[27]	2+1	2.4	1
NPLQCD 07B	[28]	2+1	2.52	2
NPLQCD 06B	[29]	2+1	2.5	2

Table 120: Finite-volume effects in determinations of the low-energy constants related to πK scattering.

C.4 Notes to Sec. 6 on kaon mixing

C.4.1 $K \to \pi\pi$ decay amplitudes

Collab.	Ref.	N_f	a [fm]	Description
RBC/UKQCD 20	[30]	2+1	0.143	Single lattice spacing.
RBC/UKQCD 15G	[31]	2+1	0.143	Single lattice spacing.
RBC/UKQCD 15F	[32]	2+1	0.114, 0.084	Combined chiral-continuum fit based on two values of the lattice spacing. Systematic error associated with the extrapolation to the continuum limit is negligible with respect to the statistical errors.

Table 121: Continuum extrapolations/estimation of lattice artifacts in determinations of the $K \to \pi\pi$ decay amplitudes.

Collab.	Ref.	N_f	$M_{\pi,\mathrm{min}}$ [MeV]	Description
RBC/UKQCD 20	[30]	2+1	142.3	Single pion mass value, close to the physical point.
RBC/UKQCD 15G	[31]	2+1	143.1	Single pion mass value, close to the physical point.
RBC/UKQCD 15F	[32]	2+1	139.1, 139.2	Single pion mass value—close to the physical point—at each of the two values of the lattice spacing.

Table 122: Chiral extrapolation/minimum pion mass in determinations of the $K \to \pi\pi$ decay amplitudes.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
RBC/UKQCD 20	[30]	2+1	4.6	3.3	Finite volume effects amount to a 7% systematic error contribution to the final error budget of A_0.
RBC/UKQCD 15G	[31]	2+1	4.6	3.3	Finite volume effects amount to a 7% systematic error contribution to the final error budget of A_0.
RBC/UKQCD 15F	[32]	2+1	5.5, 5.4	3.8	Finite volume effects amount to a 3.5% systematic error contribution to the error budget of $\mathrm{Im}(A_2)/\mathrm{Re}(A_2)$.

Table 123: Finite volume effects in determinations of the $K \to \pi\pi$ decay amplitudes.

Collab.	Ref.	N_f	Ren.	running match.	Description
RBC/UKQCD 20	[30]	2+1	RI	PT1ℓ	Two different RI-SMOM schemes are used. The relative systematic errors due to the renormalisation of the relevant operators amount to 4%, while those arising from the computation of the Wilson coefficients in the $\overline{\mathrm{MS}}$ scheme correspond to 12%.
RBC/UKQCD 15G	[31]	2+1	RI	PT1ℓ	Two different RI-SMOM schemes are used. The relative systematic errors due to the renormalisation of the relevant operators amount to 15%, while those arising from the computation of the Wilson coefficients in the $\overline{\mathrm{MS}}$ scheme correspond to 12%.
RBC/UKQCD 15F	[32]	2+1	RI	PT1ℓ	Two different RI-SMOM schemes are used. The relative systematic errors for the conversion to $\overline{\mathrm{MS}}$ are 2.9% for ReA_0 and 7% for ImA_0.

Table 124: Running and matching in determinations of the $K \to \pi\pi$ decay amplitudes.

C.4.2 Kaon B-parameter B_K

No new calculations w.r.t. the previous FLAG report.

C.4.3 Kaon BSM B-parameters

No new calculations w.r.t. the previous FLAG report.

C.5 Notes to Sec. 7 on D-meson decay constants and form factors

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
χQCD 20A	[33]	2+1	114	Simulations are performed very close to the physical point and linear interpolations/extrapolations are used to correct for mismatches.
RBC/UKQCD 17, RBC/UKQCD 18A	[34, 35]	2+1	139, 139, 232	The lattice spacing, pion-mass and charm-quark mass dependences are fit simultaneously through a Taylor expansion in a^2, $(m_\pi^2 - m_\pi^{2phys})$ and $1/m_H - 1/m_{D_{(s)}}$.

Table 125: Chiral extrapolation/minimum pion mass in $N_f = 2+1$ determinations of the D- and D_s-meson decay constants. For actions with multiple species of pions, masses quoted are the RMS pion masses. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
Blossier 18, Balasubramanian 19	[36, 37]	2	282, 194, 269	Linear fits (in m_π^2 and in a^2) are used in the combined chiral/continuum extrapolation. NLO HMχPT expressions are used for a cross-check, concluding however that there are not enough data points to be sensitive to the NLO terms.

Table 126: Chiral extrapolation/minimum pion mass in $N_f = 2$ determinations of the D- and D_s-meson decay constants. For actions with multiple species of pions, masses quoted are the RMS pion masses. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$	Description
χQCD 20A	[33]	2+1	5.5	3.2	No explicit discussion of FSE.
RBC/UKQCD 17, RBC/UKQCD 18A	[34, 35]	2+1	5.5/2.7, 5.4/2.7, 3.4	3.86, 3.78, 4.11	FV errors estimated to be at the permille level by either comparing values of $m_\pi L$ to the study of FSE by MILC in [38] or by 1-loop HMχPT.

Table 127: Finite-volume effects in $N_f = 2+1$ determinations of the D- and D_s-meson decay constants. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest masses are quoted.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
Blossier 18, Balasubramanian 19	[36, 37]	2	2.4/3.6, 2.1/3.1/4.2, 2.3/3.1	5.2, 4.1, 4.2	No explicit discussion of FV effects, but $m_\pi L > 4$ always.

Table 128: Finite-volume effects in $N_f = 2$ determinations of the D- and D_s-meson decay constants. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest masses are quoted.

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale Setting
χQCD 20A	[33]	2+1	0.115	Cutoff effects are estimated to be 2% by comparing the result for f_{D_s} with the one obtained in [39].	m_Ω used for scale setting by RBC/UKQCD, which produced the ensemble
RBC/UKQCD 17, RBC/UKQCD 18A	[34, 35]	2+1	0.11, 0.08, 0.07	The lattice spacing, pion-mass and charm-quark mass dependences are fit simultaneously through a Taylor expansion in a^2, $(m_\pi^2 - m_\pi^{2phys})$ and $1/m_H - 1/m_{D_{(s)}}$.	The lattice scale and physical light-quark masses have been determined using m_π, m_K and m_Ω as inputs.

Table 129: Lattice spacings and description of actions used in $N_f = 2 + 1$ determinations of the D- and D_s-meson decay constants.

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale Setting
Blossier 18, Balasubramanian 19	[36, 37]	2	0.075, 0.065, 0.048	Linear fits (in m_π^2 and in a^2) are used in the combined chiral/continuum extrapolation.	Scale set through f_K.

Table 130: Lattice spacings and description of actions used in $N_f = 2$ determinations of the D- and D_s-meson decay constants.

Collab.	Ref.	N_f	Ren.	Description
χQCD 20A	[33]	2+1	RI	The decay constants are extracted from an exact lattice Ward identity or from NP renormalized operators.
RBC/UKQCD 17, RBC/UKQCD 18A	[34, 35]	2+1	mNPR	The local current is renormalized nonperturbatively for the case of the unmixed action; however in the actual computation the domain wall height is chosen differently in the valence than in the sea and the effect of that on the renormalization constant is estimated to be 0.4% through a study in the RI/SMOM scheme.

Table 131: Operator renormalization in $N_f = 2 + 1$ determinations of the D- and D_s-meson decay constants.

Collab.	Ref.	N_f	Ren.	Description
Blossier 18, Balasubramanian 19	[36, 37]	2	SF	NP renormalization and improvement of the axial current (am terms included at 1-loop).

Table 132: Operator renormalization in $N_f = 2$ determinations of the D- and D_s-meson decay constants.

Collab.	Ref.	N_f	Action	Description
χQCD 20A	[33]	2+1	Overlap on DW	$am_c \approx 0.73$ set by using m_{D_s}.
RBC/UKQCD 17, RBC/UKQCD 18A	[34, 35]	2+1	Möbius-DWF on Shamir-DWF or Möbius-DWF	$0.18 < am_h < 0.4$. Charm discretization errors estimated using different ways to define the charm quark mass (through D, D_s or η_c) in the global fits.

Table 133: Heavy-quark treatment in $N_f = 2 + 1$ determinations of the D- and D_s-meson decay constants.

Collab.	Ref.	N_f	Action	Description
Blossier 18, Balasubramanian 19	[36, 37]	2	npSW	$am_c \leq 0.32$. Axial current nonperturbatively improved (O(am) at 1-loop).

Table 134: Heavy-quark treatment in $N_f = 2$ determinations of the D- and D_s-meson decay constants.

C.5.1 Form factors for semileptonic decays of charmed hadrons

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale setting
HPQCD 21A	[40]	2+1+1	0.042, 0.06, 0.09, 0.12, 0.15	Modified z-expansion fit combining the continuum and chiral extrapolations and the momentum-transfer dependence. Discretization effects assumed dominated by the charm scale. Discretization errors on form factors between 0.4% and 1.2% as a function of the momentum transfer.	Scale setting from f_π via the flow quantity w_0 [41–43].
Zhang 21	[44]	2+1	0.080, 0.11	Continuum extrapolation combined with fit to q^2-dependence of form factors in a "modified" z-expansion. Systematics estimated from difference between extrapolated results and results at smallest lattice spacing, and difference between two current renormalization methods.	Set from Wilson-flow quantity w_0.
HPQCD 20	[45]	2+1+1	0.06, 0.09, 0.12, 0.15	Modified z-expansion fit combining the continuum and chiral extrapolations and the momentum-transfer dependence, and, for the heavy-HISQ spectator b quark, the dependence on $1/m_Q$. The analysis combines data with NRQCD b quarks and data with HISQ heavy quarks.	Scale setting from f_π via the flow quantity w_0 [41–43].

Table 135: Continuum extrapolations/estimation of lattice artifacts in $N_f = 2+1+1$ determinations of form factors for semileptonic decays of charmed hadrons.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
HPQCD 21A	[40]	2+1+1	315, 329, 129, 132, 131	Modified z-expansion fit combining the continuum and chiral extrapolations and the momentum-transfer dependence. Polynomial dependence on quark masses, supplemented by a pion chiral logarithm. Fit result compared with alternative approach based on cubic splines in q^2.
Zhang 21	[44]	2+1	300, 290	Dependence on pion mass neglected. No estimate of resulting systematic uncertainty.
HPQCD 20	[45]	2+1+1	329, 316, 132/305, 131/305	Modified z-expansion fit combining the continuum and chiral extrapolations and the momentum-transfer dependence, and, for the heavy-HISQ spectator b quark, the dependence on $1/m_Q$. The analysis combines data with NRQCD b quarks and data with HISQ heavy quarks.

Table 136: Chiral extrapolation/minimum pion mass in determinations of form factors for semileptonic decays of charmed hadrons. For actions with multiple species of pions, masses quoted are the RMS pion masses for $N_f = 2+1$ and the Goldstone mode mass for $N_f = 2+1+1$. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min} L$	Description
HPQCD 21A	[40]	2+1+1	2.73, 2.72, 2.81/5.62, 2.93/5.87, 2.45/4.89	$\gtrsim 3.7$	Finite volume correction included in chiral fit, claimed to be a negligible effect. Effect of frozen topology in finest ensemble not discussed.
Zhang 21	[44]	2+1+1	2.6, 2.6	$\gtrsim 3.8$	No discussion of finite-volume effects.
HPQCD 20	[45]	2+1+1	2.72, 2.81, 2.93/5.87, 2.45/4.89	$\gtrsim 3.8$	Physical point ensemble at $a \simeq 0.15$ fm has $m_\pi L = 3.3$; the statement $m_\pi L \gtrsim 3.8$ applies to the other five ensembles.

Table 137: Finite-volume effects in determinations of form factors for semileptonic decays of charmed hadrons. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest pion masses are quoted.

Collab.	Ref.	N_f	Ren.	Description
HPQCD 21A	[40]	2+1+1	NP	Vector current normalized by imposing Ward identity at zero recoil.
Zhang 21	[44]	2+1	NP	Local vector current renormalized using ratio to conserved vector current. Axial current renormalized using ratio of off-shell quark matrix elements.
HPQCD 20	[45]	2+1+1	NP	Vector current normalized by imposing Ward identity at zero recoil.

Table 138: Operator renormalization in determinations of form factors for semileptonic decays of charmed hadrons.

Collab.	Ref.	N_f	Action	Description
HPQCD 21A	[40]	2+1+1	HISQ	Bare charm-quark mass $0.194 \lesssim am_c \lesssim 0.8605$.
Zhang 21	[44]	2+1+1	SW	Bare charm-quark mass $0.235 \lesssim am_c \lesssim 0.485$. No $\mathcal{O}(a)$ improvement of currents.
HPQCD 20	[45]	2+1+1	Charm: HISQ Bottom (spectator): HISQ and NRQCD	Bare charm-quark HIQS mass $0.274 \lesssim am_c \lesssim 0.827$. Bare bottom-quark HIQS mass $0.274 \lesssim am_b \lesssim 0.8$.

Table 139: Heavy-quark treatment in determinations of form factors for semileptonic decays of charmed hadrons.

C.6 Notes to Sec. 8 on B-meson decay constants, mixing parameters and form factors

C.6.1 $B_{(s)}$-meson decay constants

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
RBC/UKQCD 18A	[35]	2+1	139,139,232	Three or four light-quark masses per lattice spacing except for the finest lattice spacing. Generic fits to $m_\pi^2 - (m_\pi^{\rm phys})^2$ and a^2 in the combined chiral-continuum extrapolation, with systematic errors estimated to be from 0.3% to 0.5% in f_{B_s}/f_B.

Table 140: Chiral extrapolation/minimum pion mass in determinations of the B- and B_s-meson decay constants for $N_f = 2+1$ simulations. For actions with multiple species of pions, masses quoted are the RMS pion masses. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
Balasubramamian 19	[37]	2	282, 194, 269	Two pion masses for the coarsest and finest lattice spacings, and four pion masses for the intermediate lattice spacing. Generic fits to m_π^2 and a^2 in the combined continuum-chiral extrapolation, with systematic effects obtained by including a generic NLO chiral term $m_\pi^2 \log m_\pi^2$. Detailed error budget not provided, but total systematic uncertainty estimated as 1.1-1.2%.

Table 141: Chiral extrapolation/minimum pion mass in determinations of the B- and B_s-meson decay constants for $N_f = 2$ simulations. For actions with multiple species of pions, masses quoted are the RMS pion masses (where available). The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$	Description
RBC/UKQCD 18A	[35]	2+1	2.65/5.47, 2.65/5.35, 3.49	3.86, 3.78, 4.11	Finite-volume effects are estimated to be 0.18% in f_{B_s}/f_B.

Table 142: Finite-volume effects in determinations of the B- and B_s-meson decay constants for $N_f = 2+1$ simulations. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
Balasubramamian 19	[37]	2	2.4/3.6, 2.1/3.1/4.2, 2.3/3.1	4.1, 4.1, 4.0	No explicit estimate of FV errors, but expected to be much smaller than other uncertainties.

Table 143: Finite-volume effects in determinations of the B- and B_s-meson decay constants for $N_f = 2$ simulations. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings.

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale setting
RBC/UKQCD 18A	[35]	2+1	0.11, 0.084, 0.073	Combined continuum and chiral extrapolation with linear in a^2 term. Systematic errors estimated to be from 0.3% to 0.5% in f_{B_s}/f_B.	Scale set by the Ω baryon mass. No estimate for scale uncertainty, but expected to be negligible in f_{B_s}/f_B.

Table 144: Continuum extrapolations/estimation of lattice artifacts in determinations of the B and B_s meson decay constants for $N_f = 2+1$ simulations.

Collab.	Ref.	N_f	Ren.	Description
RBC/UKQCD 18A	[35]	2+1	–	Mass-independent operator renormalization cancels in f_{B_s}/f_B.

Table 145: Description of the renormalization/matching procedure adopted in the determinations of the B- and B_s-meson decay constants for $N_f = 2+1$ simulations.

Collab.	Ref.	N_f	Ren.	Description
Balasubramamian 19	[37]	2	PT2ℓ	Perturbative coefficient relating HQET and QCD currents included at NNLO in continuum perturbation theory. Systematic uncertainties estimated using fits including only NLO coefficients.

Table 146: Description of the renormalization/matching procedure adopted in the determinations of the B- and B_s-meson decay constants for $N_f = 2$ simulations.

Collab.	Ref.	N_f	Action	Description
RBC/UKQCD 18A	[35]	2+1	DWF	Linear extrapolation in inverse heavy quark mass $1/M_H$, with systematic errors estimated, by excluding heaviest and lightest valence quark masses, to be 0.47% in f_{B_s}/f_B. HQ discretization effects are estimated to be 0.01% inf_{B_s}/f_B.

Table 147: Heavy-quark treatment in $N_f = 2+1$ determinations of the B-and B_s-meson decay constants.

Collab.	Ref.	N_f	Action	Description
Balasubramamian 19	[37]	2	Wilson	Expansion in inverse of pseudoscalar heavy-strange meson mass, $1/M_{H_s}$. HQ discretization effects included in generic fit linear in $a^2(aM_{H_s})^2$. Detailed error budget not provided, but systematic uncertainty estimated as 1.1-2.3% in f_{B_s}.

Table 148: Heavy-quark treatment in $N_f = 2$ determinations of the B-and B_s-meson decay constants.

C.6.2 $B_{(s)}$-meson mixing matrix elements

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale setting
HPQCD 19A	[46]	2+1+1	0.15, 0.12, 0.09	Discretization errors start from $\alpha_s a^2$ and are included in the systematic error. It is estimated as 1.8% for individual bag parameters. Residual $\alpha_s a^2$ and a^4 errors from wrong-spin contributions are subtracted by including them in the chiral fit.	Scale setting done using Υ and Υ' mass splitting [47].

Table 149: Continuum extrapolations/estimation of lattice artifacts in determinations of the neutral B-meson mixing matrix elements for $N_f = 2+1+1$ simulations.

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale setting
RBC/UKQCD 18A	[35]	2+1	0.11, 0.08, 0.07	Combined continuum (a^2) and heavy quark ($1/m_H$) extrapolation with the LO pion mass dependence (m_π^2) in the global fit.	Lattice scale and target quark masses are set using Ω, K and π masses [34, 48, 49].

Table 150: Continuum extrapolations/estimation of lattice artifacts in determinations of the neutral B-meson mixing matrix elements for $N_f = 2+1$ simulations.

Collab.	Ref.	N_f	$M_{\pi,\min}$ [MeV]	Description
HPQCD 19A	[46]	2+1+1	311, 241, –	Pion mass in the Goldston channel is as small as 130 MeV for two coarser lattices. NLO HMrSχPT is used with NNLO analytic terms and other discretization errors. Staggered wrong-spin contributions are included.
RBC/UKQCD 18A	[35]	2+1	139, 139, 234	Combined continuum (a^2) and heavy quark ($1/m_H$) extrapolation with the LO pion mass dependence (m_π^2) in the global fit.

Table 151: Chiral extrapolation/minimum pion mass in determinations of the neutral B-meson mixing matrix elements. For actions with multiple species of pions, masses quoted are the RMS pion masses (where available). The different $M_{\pi,\min}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$	Description
HPQCD 19A	[46]	2+1+1	2.4/3.5/4.6, 2.9/3.8/5.7, 2.8	7.3, 7.0, -	FV error is estimated to be negligible from FV HMχPT.
RBC/UKQCD 18A	[35]	2+1	2.7/5.5, 2.6/5.3, 3.5	3.9, 3.8, 4.0	FV error is estimated to be less than 0.18% for $SU(3)$-breaking ratios from FV HMχPT.

Table 152: Finite-volume effects in determinations of the neutral B-meson mixing matrix elements. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, masses quoted are the RMS pion masses (where available).

Collab.	Ref.	N_f	Ren.	Description
HPQCD 19A	[46]	2+1+1	PT1ℓ	HISQ-NRQCD 4-quark operators are matched through $O(1/M)$ and renormalized to 1-loop: included are those of $O(\alpha_s)$, $O(\Lambda_{\rm QCD}/M)$, $O(\alpha_s/aM)$, $O(\alpha_s\,\Lambda_{\rm QCD}/M)$. Remnant error is dominated by $O(\alpha_s\Lambda_{\rm QCD}/M)$ 2.9% and $O(\alpha_s^2)$ 2.1% for individual bag parameters. Associated error for their SU(3) breaking ratio are negligible.
RBC/UKQCD 18A	[35]	2+1	-	Operators are renormalized multiplicatively due to chiral symmetry of DWF. No need to calculate the renormalization factor since only the $SU(3)$ breaking ratios are examined.

Table 153: Operator renormalization in determinations of the neutral B-meson mixing matrix elements.

Collab.	Ref.	N_f	Action	Description
HPQCD 19A	[46]	2+1+1	NRQCD	See the entry in Tab. 153.
RBC/UKQCD 18A	[35]	2+1	DWF	Domain-wall fermion with 3 stout-smearing extends the reach to heavy mass, allowing to simulate up to half of the b-quark mass. Heavy mass errors on ξ are estimated as 0.8% from fitting range and 0.4% from higher order $(1/M^2)$ by power counting.

Table 154: Heavy-quark treatment in determinations of the neutral B-meson mixing matrix elements.

C.6.3 Form factors entering determinations of $|V_{ub}|$ ($B \to \pi\ell\nu$, $B_s \to K\ell\nu$, $\Lambda_b \to p\ell\bar{\nu}$)

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale setting
RBC/UKQCD 23	[50]	2+1	0.071, 0.083, 0.11	Joint chiral-continuum extrapolation using $SU(2)$ hard-pion HMχPT. Systematic uncertainty estimated by varying fit ansatz and form of coefficients, as well as implementing different cuts on data.	Scale implicitly set in the light-quark sector using the Ω^- mass, cf. [34, 35, 49].
JLQCD 22	[51]	2+1	0.044, 0.055, 0.080	Discretization effects treated using overall factors of $(1 + C_{a^2}(\Lambda_{\rm QCD}a)^2 + C_{(am_Q)^2}(am_Q)^2)$, with independent coefficients for the two form factors. Systematic uncertainties estimated by adding $C_{a^4}(\Lambda_{\rm QCD}a)^4$ or $C_{(am_Q)^4}(am_Q)^4)$ terms.	Relative scale set using gradient-flow time $t_0^{1/2}/a$. Absolute scale $t_0^{1/2}$ taken from Ref. [52].
FNAL/MILC 19	[53]	2+1	0.06, 0.09, 0.12	HMrSχPT expansion used at next-to-leading order in $SU(2)$ and leading order in $1/M_B$, including next-to-next-to-leading-order (NNLO) analytic and generic discretization terms. Hard kaons assumed to decouple. Systematic uncertainties estimated by varying fit ansatz and data range. The (stat + chiral extrap + HQ discretization + g_π) uncertainty dominates the error budget, ranging from 2–3% at $q^2 \gtrsim 21$ GeV^2 to up to 8-10% in the lower end of the accessed q^2 interval.	Relative scale r_1/a set from the static-quark potential. Absolute scale r_1, including related uncertainty estimates, taken from [54].

Table 155: Continuum extrapolations/estimation of lattice artifacts in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
RBC/UKQCD 23	[50]	2+1	268, 301, 340	Joint chiral-continuum extrapolation using $SU(2)$ hard-pion HMχPT. Systematic uncertainty estimated by varying fit ansatz and form of coefficients, as well as implementing different cuts on data.
JLQCD 22	[51]	2+1	300, 300, 230	Chiral extrapolation uses $SU(2)$ hard-pion heavy-meson chiral perturbation theory at next-to-leading order. Systematic uncertainty estimated by adding M_π^4 terms or by making the coefficients of the chiral logs fit parameters.
FNAL/MILC 19	[53]	2+1	255, 277, 456	HMrSχPT expansion used at next-to-leading order in $SU(2)$ and leading order in $1/M_B$, including next-to-next-to-leading-order (NNLO) analytic and generic discretization terms. Hard kaons assumed to decouple. Systematic uncertainties estimated by varying fit ansatz and data range.

Table 156: Chiral extrapolation/minimum pion mass in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors. For actions with multiple species of pions, masses quoted are the RMS pion masses. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\rm min}L$	Description
RBC/UKQCD 23	[50]	2+1	3.4, 2.7, 2.6	4.6, 4.0, 4.4	FV effects removed by correction to chiral logs due to sums over discrete momenta; quoted maximum corrections are 0.13% for f_+ and 0.06% for f_0.
JLQCD 22	[51]	2+1	2.6, 3.9	$\gtrsim 4.0$	FV effects in form factors deemed negligible. Bias in pion mass due to topology freezing at finest lattice spacing estimated to be $\sim 0.1\%$.
FNAL/MILC 19	[53]	2+1	3.8, 2.5/2.9/3.6/5.8, 2.9	$\gtrsim 3.8$	FV effects estimated by comparing infinite volume integrals with finite sums in HMrSχPT, found to be negligible.

Table 157: Finite-volume effects in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest masses are quoted.

Collab.	Ref.	N_f	Ren.	Description
RBC/UKQCD 23	[50]	2+1	mNPR	Perturbative truncation error estimated as full size of $O(\alpha_s)$ correction at the 0.083 fm lattice spacing.
JLQCD 22	[51]	2+1	NPR	$Z_{V_{qq}}$ obtained using position-space current-current correlators. For heavier quark masses, $\sqrt{Z_{V_{QQ}} Z_{V_{qq}}}$ is used, where $Z_{V_{QQ}}$ is the renormalization factor of the flavor-conserving temporal vector current, determined using charge conservation.
FNAL/MILC 19	[53]	2+1	mNPR	Perturbative truncation error estimated at 1% with size of 1-loop correction on next-to-finest ensemble.

Table 158: Operator renormalization in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors.

Collab.	Ref.	N_f	Action	Description
RBC/UKQCD 23	[50]	2+1	Columbia RHQ	Heavy-quark discretization errors estimated by power counting.
JLQCD 22	[51]	2+1	DWF	Bare heavy-quark masses satisfy $am_Q < 0.7$ and reach from the charm mass up to 2.44 times the charm mass. Form factors extrapolated linearly in $1/m_Q$ to the bottom mass.
FNAL/MILC 19	[53]	2+1	Fermilab	(See comments for continuum limit extrapolation.)

Table 159: Heavy-quark treatment in determinations of $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, and $\Lambda_b \to p\ell\bar{\nu}$ form factors.

C.6.4 Form factors for rare decays of beauty hadrons

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale setting
Meinel 20	[55]	2+1	0.0828(3), 0.1106(3)	Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematic uncertainty estimated by repeating fit with added higher-order terms.	Scale setting using Ω mass in Ref. [49].

Table 160: Continuum extrapolations/estimation of lattice artifacts in determinations of form factors for rare decays of beauty hadrons.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
Meinel 20	[55]	2+1	303, 340	Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematic uncertainty estimated by repeating fit with added higher-order terms.

Table 161: Chiral extrapolation/minimum pion mass in determinations of form factors for rare decays of beauty hadrons. For actions with multiple species of pions, masses quoted are the RMS pion masses. The different $M_{\pi,\text{min}}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\text{min}}L$	Description
Meinel 20	[55]	2+1	2.7, 2.7	4.1, 4.6	FV effects not quantified. Effects from unstable $\Lambda^*(1520)$ not quantified.

Table 162: Finite-volume effects in determinations of form factors for rare decays of beauty hadrons. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest masses are quoted.

Collab.	Ref.	N_f	Ren.	Description
Meinel 20	[55]	2+1	mNPR	Residual matching factors ρ computed at one-loop for vector and axial-vector currents, but at tree-level only for tensor currents. A systematic uncertainty is assigned to $\rho_{T^{\mu\nu}}$ as the double of $\max(\lvert\rho_{A^\mu}-1\rvert, \lvert\rho_{V^\mu}-1\rvert)$.

Table 163: Operator renormalization in determinations of form factors for rare decays of beauty hadrons.

Collab.	Ref.	N_f	Action	Description
Meinel 20	[55]	2+1	Columbia RHQ	Discretization errors discussed as part of combined chiral-continuum-w fit. Higher-order fit also includes $\mathcal{O}(\alpha_s a\lvert\mathbf{p}\rvert)$ terms to account for missing radiative corrections to $\mathcal{O}(a)$ improvement of the currents.

Table 164: Heavy-quark treatment in determinations of form factors for rare decays of beauty hadrons.

C.6.5 Form factors entering determinations of $|V_{cb}|$ ($B_{(s)} \to D^{(*)}_{(s)}\ell\nu$, $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$) and $R(D_{(s)})$

Collab.	Ref.	N_f	a [fm]	Continuum extrapolation	Scale setting
Meinel 21	[56]	2+1	0.0828(3), 0.1106(3)	Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematics estimated by varying fit form.	Scale setting using Ω mass in Ref. [49].
HPQCD 19	[57]	2+1+1	0.044, 0.059, ≈0.09	Combined chiral-continuum extrapolation. Fractional error from continuum limit and extrapolation to physical b mass at zero recoil quoted as 1.20%.	Scale setting from f_π via the flow quantity w_0 [41–43].
HPQCD 19B	[58]	2+1+1	0.044, 0.059, ≈0.09	Combined chiral-continuum extrapolation. Fractional error from continuum limit and extrapolation to physical b mass at zero recoil quoted as 0.73% and 0.69%, respectively.	Scale setting from f_π via the flow quantity w_0 [41–43].
FNAL/MILC 22	[59]	2+1	0.045, 0.06, 0.09, 0.12, 0.15	Combined chiral-continuum extrapolation using HMrSχPT. Total uncertainty quoted at 0.7%.	Relative scale r_1/a set from the static-quark potential. Absolute scale r_1, including related uncertainty estimates, taken from [54]. Uncertainty related to scale setting estimated at less than 0.1%.

Table 165: Continuum extrapolations/estimation of lattice artifacts in $N_f = 2+1$ determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$.

Collab.	Ref.	N_f	$M_{\pi,\rm min}$ [MeV]	Description
Meinel 21	[56]	2+1	303, 340	Combined chiral-continuum extrapolation as part of the expansion of form factor shape in powers of $w-1$. Systematic uncertainty estimated by repeating fit with added higher-order terms.
HPQCD 19	[57]	2+1+1	315, 329, 129 [60]	Combined chiral-continuum extrapolation using rSχPT. No specific uncertainty coming from chiral extrapolation quoted.
HPQCD 19B	[58]	2+1+1	315, 329, 129 [60]	Combined chiral-continuum extrapolation using rSχPT. No specific uncertainty coming from chiral extrapolation quoted.
FNAL/MILC 22	[59]	2+1	320, 220, 180, 270, 340	Combined chiral-continuum extrapolation using HMrSχPT. Systematic errors estimated by adding higher-order analytic terms and varying the D^*-D-π coupling. Total uncertainty quoted at 0.7%.

Table 166: Chiral extrapolation/minimum pion mass in $N_f = 2+1$ determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$. For actions with multiple species of pions, masses quoted are the RMS pion masses for $N_f = 2+1$ and the Goldstone mode mass for $N_f = 2+1+1$. The different $M_{\pi,\rm min}$ entries correspond to the different lattice spacings.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
Meinel 21	[56]	2+1	2.7, 2.7	4.1, 4.6	FV effects not quantified. Effects from unstable Λ_c^* not quantified.
HPQCD 19	[57]	2+1+1	2.73, 2.72, 2.81/5.62	4.3, 4.5, 3.7/4.5, 3.9/4.5, 3.3/3.8	FV effects neglected. This includes the effect of frozen topology in the finest ensemble.
HPQCD 19B	[58]	2+1+1	2.73, 2.72, 2.81/5.62	4.3, 4.5, 3.7/4.5, 3.9/4.5, 3.3/3.8	FV effects neglected. This includes the effect of frozen topology in the finest ensemble.
FNAL/MILC 22	[59]	2+1	4.6, 4.3–6.3, 4.1–5.8, 3.8–6.2, 3.9	$\gtrsim 3.8$	FV error estimated to be negligible.

Table 167: Finite-volume effects in determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$. Each L-entry corresponds to a different lattice spacing, with multiple spatial volumes at some lattice spacings. For actions with multiple species of pions, the lightest pion masses are quoted.

Collab.	Ref.	N_f	Ren.	Description
Meinel 21	[56]	2+1	mNPR	Residual matching factors ρ computed at one-loop for vector and axial-vector currents, but at tree-level only for tensor currents. A systematic uncertainty is assigned to $\rho_{T^{\mu\nu}}$ as the double of $\max(\lvert\rho_{A^\mu} - 1\rvert, \lvert\rho_{V^\mu} - 1\rvert)$.
HPQCD 19	[57]	2+1+1	NP	Currents normalized non-perturbatively by imposing that $f_+(0) = f_0(0) = 1$ for transitions involving initial and final mesons with identical masses.
HPQCD 19B	[58]	2+1+1	NP	Currents renormalized nonperturbatively using PCAC relation.
FNAL/MILC 22	[59]	2+1	mNPR	Majority of current renormalization factor cancels in double ratio of lattice correlation functions. Remaining correction calculated with 1-loop tadpole-improved lattice perturbation theory. Systematic uncertainty estimated at 0.1%.

Table 168: Operator renormalization in determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$.

Collab.	Ref.	N_f	Action	Description
Meinel 21	[56]	2+1	Columbia RHQ for both the b and c quarks.	Discretization errors discussed as part of combined chiral-continuum-w fit. Higher-order fit also includes $\mathcal{O}(\alpha_s a\vert\mathbf{p}\vert)$ terms to account for missing radiative corrections to $\mathcal{O}(a)$ improvement of the currents.
HPQCD 19	[57]	2+1+1	HISQ for both the b and c quarks.	Values of bare heavy quark masses up to $am_h = 0.8$. Fractional error from continuum limit and extrapolation to physical b mass at zero recoil quoted as 1.20%.
HPQCD 19B	[58]	2+1+1	HISQ for both the b and c quarks.	Values of bare heavy-quark masses up to $am_h = 0.8$. Fractional error from extrapolation to physical b mass quoted as 0.69%.
FNAL/MILC 22	[59]	2+1	Fermilab	Discretization errors discussed as part of combined chiral-continuum stemming from $\alpha_s a$, a^2 and a^3 terms.

Table 169: Heavy-quark treatment in determinations of $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$ and $\Lambda_b \to \Lambda_c^{(*)}\ell\bar{\nu}$ form factors, and of $R(D_{(s)})$.

C.7 Notes to Sec. 9 on the strong coupling α_s

In this section we provide more detailed information on the simulations used to calculate the strong coupling α_s. We present this information only for results that have appeared since FLAG 19. For information on previous calculations not listed here we refer the previous reports FLAG 19 [1] and FLAG 16 [2].

C.7.1 Renormalization scale and perturbative behaviour

Collab.	Ref.	N_f	α_{eff}	n_l	Description
Nada 20	[61]	0	SF: 0.08–0.15 GF: 0.08–0.24	2	step-scaling with GF, non-pert matching to SF, same gauge configurations as in [62]
Husung 20	[63]	0	0.17–0.36	3	estimated from α_{qq}
Dalla Brida 19	[62]	0	SF: 0.08–0.15 GF: 0.08–0.95	2	step-scaling with GF, non-pert matching to SF

Table 170: Renormalization scale and perturbative behaviour of α_s determinations for $N_f = 0$.

Collab.	Ref.	N_f	α_{eff}	n_l	Description
Petreczky 20	[64]	2+1	0.22–0.38	2	$m_h = m_c - 4m_c$
Ayala 20	[65]	2+1	0.2–0.4	3	$1/r > 2$ GeV
Cali 20	[66]	2+1	0.235–0.308	3	$\lvert x \rvert$ =0.13–0.19 fm
Boito 20	[67]	2+1	0.38	2	only $m_h = m_c$ is used
TUMQCD 19	[68]	2+1	0.2–0.4	3	$r < 0.073$ fm
Zafeiropoulos 19	[69]	2+1	0.35 - 0.42	3	α_T for $p \sim$3.0–3.7 GeV
Petreczky 19	[8]	2+1	0.31, 0.38	2	only results for $m_h = m_c, 1.5m_c$ are reviewed

Table 171: Renormalization scale and perturbative behaviour of α_s determinations for $N_f = 3$.

C.7.2 Continuum limit

Collab.	Ref.	N_f	$a\,\mu$	Description
Nada 20	[61]	0	GF: $0.125 < a\mu < 0.042$ SF: $0.17 < a\mu < 0.063$	step scaling GF scheme, non-pert matching to SF scheme
Husung 20	[63]	0	$a\mu < 0.57$	only $r/a > 3/5$ are used in the analysis
Dalla Brida 19	[62]	0	GF: $0.125 < a\mu < 0.042$ SF: $0.17 < a\mu < 0.063$	step scaling GF scheme, non-pert matching to SF scheme

Table 172: Continuum limit for α_s determinations with $N_f = 0$.

Collab.	Ref.	N_f	$a\,\mu$	Description
Petreczky 20	[64]	2+1	$a\mu = 0.32 - 5.40$	$a = 0.025 - 0.104$ fm, $m_h = m_c - 4m_c$
Ayala 20	[65]	2+1	$a\mu < 0.71$	only $r/a > \sqrt{8}$ are considered
Cali 20	[66]	2+1	$a\mu = 0.21 - 0.58$	$a = 0.039 - 0.076$ fm
Boito 20	[67]	2+1	NA	previously published continuum results were used
TUMQCD 19	[68]	2+1	$a\mu < 2$	r/a values down to 1 are used but it is checked that the result does not change if only $r/a > \sqrt{8}$ are considered
Zafeiropoulos 19	[69]	2+1	$a\mu \sim 1.3 - 1.6$	Two lattice spacings $a = 0.11, 0.08\,\text{fm}$ $\alpha_{\text{eff}} = 0.3$ not reached.
Petreczky 19	[8]	2+1	$a\mu < 0.49$	only results for $m_h = m_c, 1.5m_c$ are reviewed

Table 173: Continuum limit for α_s determinations with $N_f = 3$.

C.8 Notes to Sec. 10 on nucleon matrix elements

Collab.	Ref.	N_f	a [fm]	Description
CalLat 19	[70]	2+1+1	0.15,0.12,0.09	Extrapolation to the physical point via simultaneous fit in the lattice spacing, M_π and $M_\pi L$, including terms of order a^2 and a^4.
ETM 19	[71]	2+1+1	0.08	Single lattice spacing.

Table 174: Continuum extrapolations/estimation of lattice artifacts in determinations of the isovector axial, scalar and tensor charges with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	a [fm]	Description
χQCD 21	[72]	2+1	0.06,0.08,0.11,0.14	Extrapolation performed using a quadratic term in a as part of a simultaneous fit in a, M_π and $M_\pi L$.
NME 21	[73]	2+1	0.07,0.09,0.13	Extrapolation performed using a linear term in a as part of a simultaneous fit in a, M_π and $M_\pi L$.
RBC/UKQCD 19	[74]	2+1	0.14	Single lattice spacing.
LHPC 19	[75]	2+1	0.09,0.12	No extrapolation performed; final result taken from finer lattice with an enlarged error bar to account for result at coarser a.
Mainz 19	[76]	2+1	0.05,0.06,0.08,0.09	Extrapolation performed as part of a simultaneous fit in a, M_π and $M_\pi L$.
PACS 18A	[77]	2+1	0.085	Single lattice spacing.

Table 175: Continuum extrapolations/estimation of lattice artifacts in determinations of the isovector axial, scalar and tensor charges with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
CalLat 19	[70]	2+1+1	135,130,220	Fit performed including analytic and non-analytic terms in M_π up to order M_π^4.
ETM 19	[71]	2+1+1	139	Single pion mass within 3% of the physical value.

Table 176: Chiral extrapolation/minimum pion mass in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1 + 1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
χQCD 21	[72]	2+1	139, 171, 302, 337, 371	Extrapolation performed using quadratic terms in M_π from partially quenched chiral perturbation theory as part of a simultaneous fit in a, M_π and $M_\pi L$.
NME 21	[73]	2+1	170,170,270,285	Extrapolation performed using a quadratic term in M_π as part of a simultaneous fit in a, M_π and $M_\pi L$.
RBC/UKQCD 19	[74]	2+1	170,250	Does not quote extrapolated value, but shows low mass dependence.
LHPC 19	[75]	2+1	133,137	Does not perform extrapolation; comparison of results at two near-physical pion masses.
Mainz 19	[76]	2+1	220, 290, 200, 260	Extrapolation performed using logarithmic and quadratic terms in M_π as part of a simultaneous fit in a, M_π and $M_\pi L$.
PACS 18A	[77]	2+1	135	Single, physical pion mass.

Table 177: Chiral extrapolation/minimum pion mass in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1$ quark flavours.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
CalLat 19	[70]	2+1+1	2.4–7.2, 2.9–5.8, 2.9–4.2	4.9, 3.8, 4.7	Fit performed including a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in a^2, M_π and $M_\pi L$.
ETM 19	[71]	2+1+1	5.12, 4.5-6.0	3.62, 2.98	Final result quoted from the $M_\pi L = 3.62$ ensemble only.

Table 178: Finite-volume effects in determinations of the isovector axial, scalar and tensor charges with $N_f = 2+1+1$ quark flavours.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
χQCD 21	[72]	2+1	4.5 5.3, 2.6, 2.6, 1.92	3.7, 3.9, 3.6, 3.8, 3.6	Extrapolation performed including a term of the form $e^{-M_\pi L}$ as part of a simultaneous fit in a, M_π and $M_\pi L$.
NME 21	[73]	2+1	4.2, 2.9-4.3, 4.3-5.8, 3.4-5.0	5.87, 4.09, 3.75, 4.28	Extrapolation performed including a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in a^2, M_π and $M_\pi L$.
RBC/UKQCD 19	[74]	2+1	4.5	4.0	Does not quote extrapolated value.
LHPC 19	[75]	2+1	5.9, 5.6	4.0, 3.9	Does not quote extrapolated value.
Mainz 19	[76]	2+1	2.8-4.1, 2.4-3.6, 2.1-4.1,2.4-3.2	4.7, 5.3, 4.2, 4.3	Extrapolation performed including a term of the form $M_\pi^2 e^{-M_\pi L}/\sqrt{M_\pi L}$ as part of a simultaneous fit in a^2, M_π and $M_\pi L$.
PACS 18A	[77]	2+1	10.8	7.4	Single spatial volume of 10.8 fm at physical pion mass $M_\pi L = 7.4$.

Table 179: Finite-volume effects in determinations of the isovector axial, scalar and tensor charges with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	Ren.
CalLat 19	[70]	2+1+1	RI-MOM
ETM 19	[71]	2+1+1	RI-MOM

Table 180: Renormalization in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1 + 1$ quark flavours.

Collab.	Ref.	N_f	Ren.
χQCD 21	[72]	2+1	RI-MOM
NME 21	[73]	2+1	RI-SMOM
RBC/UKQCD 19	[74]	2+1	RI-SMOM
LHPC 19	[75]	2+1	RI-MOM, RI-SMOM
Mainz 19	[76]	2+1	RI-MOM, SF
PACS 18A	[77]	2+1	SF

Table 181: Renormalization in determinations of the isovector axial, scalar and tensor charges with $2 + 1$ quark flavours.

Collab.	Ref.	N_f	τ [fm]	Description
CalLat 19	[70]	2+1+1	all	Two-state fits to the τ-dependence of summed operator insertion for $\tau \geq 0.3$ fm.
ETM 19	[71]	2+1+1	[0.64–1.6]	Fits to the τ- and t-dependence of three-point correlators using multiple methods (plateau, two- and three-state fits, summation), with the two-state fits quoted for the final result.

Table 182: Control of excited state contamination in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1 + 1$ quark flavours. The comma-separated list of numbers in square brackets denote the range of source-sink separations τ (in fermi) at each value of the bare coupling.

Collab.	Ref.	N_f	τ [fm]	Description
χQCD 21	[72]	2+1	[0.9–1.1] [1–1.2] [0.9–1.3,0.9–1.3] [1–1.5]	Fits to the τ- and t-dependence of three-point correlators using up to three lowest-lying states.
NME 21	[73]	2+1	[1.0–1.8] [0.75–1.3,0.75–1.7] [0.72–1.46,0.72–1.46] [0.8–1.4,0.9–1.5]	Fits to the τ- and t-dependence of three-point correlators using three lowest-lying states.
RBC/UKQCD 19	[74]	2+1	[1.0,1.3]	Result quoted from t=1.3 fm point only; consistency in results from t=1.0 fm is demonstrated.
LHPC 19	[75]	2+1	[0.4–1.4] [0.9-1.5]	Multiple analysis methods used (plateau, summation, and two-state fits to each), and combined to produce final result.
Mainz 19	[76]	2+1	[1.0–1.4,1.0–1.4,1.0–1.4] [1.0–1.5,1.0–1.5] [1.0–1.4,1.0–1.4,1.0–1.4,1.0–1.4] [1.0–1.4,1.0–1.3]	Fits to the τ- and t-dependence of correlator ratios using the two lowest-lying states.
PACS 18A	[77]	2+1	[0.9–1.4]	Average of plateau values at the three largest source-sink separations.

Table 183: Control of excited state contamination in determinations of the isovector axial, scalar and tensor charges with $N_f = 2 + 1$ quark flavours. The comma-separated list of numbers in square brackets denote the range of source-sink separations τ (in fermi) at each value of the bare coupling.

Collab.	Ref.	N_f	a [fm]	Description
ETM 19	[71]	2+1+1	0.08	Not estimated.

Table 184: Continuum extrapolations/estimation of lattice artifacts in determinations of g_A^q and g_T^q.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	Description
ETM 19	[71]	2+1+1	139	Simulate close to M_π^{phys}.

Table 185: Chiral extrapolation/minimum pion mass in determinations of g_A^q and g_T^q.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\text{min}}L$	Description
ETM 19	[71]	2+1+1	5.1	3.6	FVE are anticipated to be small based on an investigation using two near physical point $N_f = 2$ ensembles with $M_\pi L = 3.0$ and 4.0.

Table 186: Finite-volume effects in determinations of g_A^q and g_T^q.

Collab.	Ref.	N_f	Ren.
ETM 19	[71]	2+1+1	RI-SMOM

Table 187: Renormalization in determinations of g_A^q and g_T^q.

Collab.	Ref.	N_f	τ [fm]	Description
ETM 19	[71]	2+1+1	[0.6–1.6]/all	Two-state fit to all τ. A comparison is made with three-state fits, plateau fits and the summation method.

Table 188: Control of excited state contamination in determinations of g_A^q and g_T^q. The comma-separated list of numbers in square brackets denote the range of source-sink separations τ (in fermi) at each value of the bare coupling.

Collab.	Ref.	N_f	a [fm]	Description
ETM 19	[71]	2+1+1	0.08	Not estimated.

Table 189: Continuum extrapolation/estimation of lattice artifacts in direct determinations of $\sigma_{\pi N}$ and σ_s.

Collab.	Ref.	N_f	$M_{\pi,\mathrm{min}}$ [MeV]	Description
ETM 19	[71]	2+1+1	139	Simulate close to M_π^{phys}.

Table 190: Chiral extrapolation/minimum pion mass in direct determinations of $\sigma_{\pi N}$ and σ_s.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\mathrm{min}}L$	Description
ETM 19	[71]	2+1+1	5.1	3.6	FVE are found to be small from an investigation using two near physical point $N_f = 2$ ensembles with $M_\pi L = 3.0$ and 4.0.

Table 191: Finite-volume effects in direct determinations of $\sigma_{\pi N}$ and σ_s.

Collab.	Ref.	N_f	Ren.	Description
ETM 19	[71]	2+1+1	na/na	

Table 192: Renormalization for direct determinations of $\sigma_{\pi N}$ and σ_s. The type of renormalization (Ren.) is given for $\sigma_{\pi N}$ first and σ_s second. The label 'na' indicates that no renormalization is required.

Collab.	Ref.	N_f	τ [fm]	Description
ETM 19	[71]	2+1+1	[0.6–1.6]/all	Two-state fit to all τ. A comparison is made with three-state fits, plateau fits and the summation method.

Table 193: Control of excited state contamination in direct determinations of $\sigma_{\pi N}$ and σ_s. The comma-separated list of numbers in square brackets denote the range of source-sink separations τ (in fermi) at each value of the bare coupling. The range of τ for the connected (disconnected) contributions to the three-point correlation functions is given first (second). If a wide range of τ values is available this is indicated by "all" in the table.

Collab.	Ref.	N_f	a [fm]	Description
BMW 20A	[78]	1+1+1+1	0.10,0.09,0.08,0.06	Combined continuum, chiral and volume fit within an extended frequentist method. $\mathcal{O}(\alpha a)$ or $\mathcal{O}(a^2)$ terms are included. M_N used to fix the lattice spacing.
Walker-Loud 08	[79]	2+1	0.12	Not estimated.

Table 194: Continuum extrapolations/estimation of lattice artifacts in determinations of $\sigma_{\pi N}$ and σ_s from the Feynman-Hellmann method.

Collab.	Ref.	N_f	$M_{\pi,\min}$ [MeV]	Description
BMW 20A	[78]	1+1+1+1	238,200,219,199	Combined continuum, chiral and volume fit within an extended frequentist method. Terms with $M_\pi^{2,3,4}$ and M_K^2 are included and cuts of $M_\pi < 420$ MeV and $M_\pi < 360$ MeV are made.
Walker-Loud 08	[79]	2+1	294	Fit using NNLO $SU(2)$ HBχPT [80] with a zero and non-zero value for $g_{\Delta N}$. Errors on the LECs that are not free parameters determine the systematic uncertainty.

Table 195: Chiral extrapolation/minimum pion mass in determinations of $\sigma_{\pi N}$ and σ_s from the Feynman-Hellmann method.

Collab.	Ref.	N_f	L [fm]	$M_{\pi,\min}L$	Description
BMW 20A	[78]	1+1+1+1	3.2, 2.9–4.3, 2.6–3.8, 1.9–3.8	4.0, 4.3, 4.1, 4.1	Combined continuum, chiral and volume fit within an extended frequentist method. FV term $M_\pi^{1/2}L^{-3/2}e^{-M_\pi L}$ added.
Walker-Loud 08	[79]	2+1	2.5	3.7	FV effects expected to be 1% or less for the lightest pion mass based on [81].

Table 196: Finite-volume effects in determinations of $\sigma_{\pi N}$ and σ_s from the Feynman-Hellmann method.

C.9 Notes to Sec. 11 on scale setting

Collab.	Ref.	N_f	a [fm]	Description
ETM 21	[13]	2+1+1	0.069, 0.079, 0.097	
CalLat 20A	[82]	2+1+1	$[0.056, 0.082, 0.11, 0.13]$, $[0.057, 0.087, 0.12, 0.15]$	Möbius Domain-wall valence quarks on HISQ / MILC sea; lattice spacings depending on scheme.
BMW 20	[83]	1+1+1+1	0.0640, 0.0787, 0.0952, 0.1116, 0.1191, 0.1315	Staggered fermion computation with isospin breaking and QED.
ETM 20	[1057]	2+1+1	0.069, 0.079, 0.097	Wilson TM fermions at maximal twist including clover term. Note that lattice spacings are not explicitely given in [1057].
ETM 18A	[85]	2+1+1	0.08	Wilson TM fermions at maximal twist including clover term.
FNAL/MILC 17	[60]	2+1+1	0.03, 0.042, 0.06, 0.09, 0.12, 0.15	Determination of f_{p4s}.
MILC 15	[86]	2+1+1	0.06, 0.09, 0.12, 0.15	Highly improved staggered quarks.
ETM 14	[87]	2+1+1	0.089, 0.082, 0.062	Wilson TM fermions at maximal twist.
FNAL/MILC 14A	[38]	2+1+1	0.06, 0.09, 0.12, 0.15	
HPQCD 13A	[41]	2+1+1	0.15, 0.12, 0.09	
HPQCD 11B	[47]	2+1+1	0.09, 0.12, 0.15	NRQCD for b quark and HISQ for valence light quarks and MILC sea.

Table 197: Continuum extrapolations/estimation of lattice artifacts in scale determinations with $N_f = 2 + 1 + 1$ quark flavours.

Collab.	Ref.	N_f	a [fm]	Description
RQCD 22	[88]	2+1	0.098, 0.085, 0.075, 0.064 ,0.049, 0.039	NP $\mathcal{O}(a)$-improved Wilson fermions with tree-level Symanzik improved gauge action.
CLS 21	[89]	2+1	0.038, 0.049, 0.063, 0.075 0.085	NP $\mathcal{O}(a)$-improved Wilson fermions with tree-level Symanzik improved gauge action.
CLS 16	[90]	2+1	0.085, 0.065, 0.05	NP $\mathcal{O}(a)$-improved Wilson fermions with LW gauge action.
QCDSF/UKQCD 15B	[91]	2+1	0.059, 0.068, 0.074, 0.082	$(a_{\rm max}/a_{\rm min})^2 = 1.94$
χQCD 14	[39]	2+1	0.084, 0.112	Valence overlap fermions on domain wall fermion gauge configurations.
HotQCD 14	[92]	2+1	$[0.04, 0.25]$	HISQ and tree-level improved Symanzik gauge action. Several ensembles with a in the quoted range and with a single $M_\pi \approx 160\,\mathrm{MeV}$.
RBC/UKQCD 14	[49]	2+1	0.06, 0.08, 0.11, 0.14	Two versions of Domain Wall Fermions combined each with the Iwasaki gauge action.
BMW 12A	[52]	2+1	0.053, 0.065, 0.077, 0.093	Wilson fermion computation.
HotQCD 11	[93]	2+1	$[0.066, 0.25]$	HISQ and tree-level improved Symanzik gauge action. Several ensembles with a in the quoted range.
RBC/UKQCD 10A	[48]	2+1	0.087, 0.114	
MILC 10	[94]	2+1	0.045, 0.06, 0.09	Asqtad staggered quarks.

Table 198: Continuum extrapolations/estimation of lattice artifacts in scale determinations with $N_f = 2+1$ quark flavours.

Collab.	Ref.	N_f	a [fm]	Description
HPQCD 09B	[95]	2+1	0.044, 0.059, 0.085, 0.12, 0.15	NRQCD for b quark and HISQ for valence light quarks on MILC asqtad sea.
MILC 09	[96–98]	2+1	$[0.045, 0.18]$	Asqtad staggered quarks, 6 different lattice spacings in the quoted range.
PACS-CS 08	[99]	2+1	0.09	Several ensembles of Wilson clover quarks and Iwasaki gauge action at a single lattice spacing.
HPQCD 05B	[100]	2+1	0.09, 0.12, 0.17	NRQCD with tree level tadpole improved couplings on MILC asqtad sea.
Aubin 04	[101]	2+1	0.09, 0.12	Asqtad staggered quarks.

Table 198: (cntd.) Continuum extrapolations/estimation of lattice artifacts in scale determinations with $N_f = 2 + 1$ quark flavours.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	$M_\pi L$	Description
ETM 21	[13]	2+1+1	134.2	3.78	
CalLat 20A	[82]	2+1+1	130	3.9	
BMW 20	[83]	1+1+1+1	Several pion masses within ±3% of physical value.	3.0	
ETM 20	[1057]	2+1+1	135	3.5	Only indirectly inferred.
ETM 18A	[85]	2+1+1	139.8	3.6	One ensemble only.
FNAL/MILC 17	[38]	2+1+1	129	3.7	Determination of f_{4ps}, M_{4ps}.
MILC 15	[86]	2+1+1	125	3.7	Four ensembles at physical point.
ETM 14	[87]	2+1+1	211	3.19	
FNAL/MILC 14A	[38]	2+1+1	130	3.7	Determination of f_{4ps}, M_{4ps}.
HPQCD 13A	[41]	2+1+1	128	3.7	
HPQCD 11B	[47]	2+1+1	211	4.0	NRQCD Υ 2s-1s splitting and η_s as input.

Table 199: Chiral extrapolation and finite-volume effects in scale determinations with $N_f = 2+1+1$ quark flavours. We list the minimum pion mass $M_{\pi,\text{min}}$ and $M_\pi L \equiv M_{\pi,\text{min}}[L(M_{\pi,\text{min}})]_{\text{max}}$ is evaluated at the maximum value of L available at $M_\pi = M_{\pi,\text{min}}$.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	$M_\pi L$	Description
RQCD 22	[88]	2+1	127/131 200	3.51/4.05 4.14	At $m = m_{\text{symm}}$. At $\tilde{m}_s = \tilde{m}_{s,\text{phys}}$
CLS 21	[89]	2+1	134	–	No further detailed information given.
CLS 16	[90]	2+1	200	4.2	Extrapolation along line $m_u + m_d + m_s = \text{const}$.
QCDSF/UKQCD 15B	[91]	2+1	228	4.1	
χQCD 14	[39]	2+1	290	4.0	
HotQCD 14	[92]	2+1	160	4.8	HISQ staggered quarks, the pion mass quoted is M_π. Results on 20 ensembles with volumes in the range $[2.6, 6.1]$ fm, we quote $M_\pi^{min} L^{max}$.
RBC/UKQCD 14	[49]	2+1	139	3.9	
BMW 12A	[52]	2+1	131 and 120	3.92 and 3.0	
HotQCD 11	[93]	2+1	160	4.8	HISQ staggered quarks, the pion mass quoted is M_π. Results on 20 ensembles with volumes in the range $[3.2, 6.1]$ fm, we quote $M_\pi^{min} L^{max}$.
RBC/UKQCD 10A	[48]	2+1	290	4.0	
MILC 10	[94]	2+1	255	4.8	Determination from global fit, scale from f_π.

Table 200: Chiral extrapolation and finite-volume effects in scale determinations with $N_f = 2+1$ quark flavours. We list the minimum pion mass $M_{\pi,\text{min}}$ and $M_\pi L \equiv M_{\pi,\text{min}}[L(M_{\pi,\text{min}})]_{\text{max}}$ is evaluated at the maximum value of L available at $M_\pi = M_{\pi,\text{min}}$.

Collab.	Ref.	N_f	$M_{\pi,\text{min}}$ [MeV]	$M_\pi L$	Description
HPQCD 09B	[95]	2+1	211	4.0	Extension of HPQCD 05B [100].
MILC 09	[96–98]	2+1	258	4.3	Asqtad staggered quarks, the pion mass quoted is M_π^{RMS}.
PACS-CS 08	[99]	2+1	156	2.3	Clover quarks, several pion masses in the range [156, 702] MeV, single lattice spacing.
HPQCD 05B	[100]	2+1	270	3.7	NRQCD Υ 2s-1s splitting as input.
Aubin 04	[101]	2+1	253	3.8	Asqtad staggered quarks, the pion mass quoted is the Goldstone mass.

Table 200: (cntd.) Chiral extrapolation and finite-volume effects in scale determinations with $N_f = 2+1$ quark flavours. We list the minimum pion mass $M_{\pi,\text{min}}$ and $M_\pi L \equiv M_{\pi,\text{min}}[L(M_{\pi,\text{min}})]_{\text{max}}$ is evaluated at the maximum value of L available at $M_\pi = M_{\pi,\text{min}}$.

References

[1] [FLAG 19] S. Aoki et al., *FLAG Review 2019: Flavour Lattice Averaging Group (FLAG)*, *Eur. Phys. J. C* **80** (2020) 113 [1902.08191].

[2] [FLAG 16] S. Aoki et al., *Review of lattice results concerning low-energy particle physics*, *Eur. Phys. J.* **C77** (2017) 112 [1607.00299].

[3] Flavour Lattice Averaging Group (FLAG), *FLAG Review*, http://flag.unibe.ch/.

[4] [ETM 21A] C. Alexandrou et al., *Quark masses using twisted mass fermion gauge ensembles*, *Phys. Rev. D* **104** (2021) 074515 [2104.13408].

[5] [ALPHA 19] M. Bruno, I. Campos, P. Fritzsch, J. Koponen, C. Pena, D. Preti et al., *Light quark masses in $N_{\rm f} = 2+1$ lattice QCD with Wilson fermions*, *Eur. Phys. J. C* **80** (2020) 169 [1911.08025].

[6] [HPQCD 20A] D. Hatton, C.T.H. Davies, B. Galloway, J. Koponen, G.P. Lepage and A.T. Lytle, *Charmonium properties from lattice QCD+QED : Hyperfine splitting, J/ψ leptonic width, charm quark mass, and a^c_μ*, *Phys. Rev. D* **102** (2020) 054511 [2005.01845].

[7] [ALPHA 21] J. Heitger, F. Joswig and S. Kuberski, *Determination of the charm quark mass in lattice QCD with $2+1$ flavours on fine lattices*, *JHEP* **05** (2021) 288 [2101.02694].

[8] P. Petreczky and J. Weber, *Strong coupling constant and heavy quark masses in (2+1)-flavor QCD*, *Phys. Rev. D* **100** (2019) 034519 [1901.06424].

[9] [HPQCD 21] D. Hatton, C.T.H. Davies, J. Koponen, G.P. Lepage and A.T. Lytle, *Determination of $\overline{m}_b/\overline{m}_c$ and $\overline{m}_b$ from $n_f = 4$ lattice QCD+QED*, *Phys. Rev. D* **103** (2021) 114508 [2102.09609].

[10] [FNAL/MILC 18] A. Bazavov et al., *$|V_{us}|$ from $K_{\ell 3}$ decay and four-flavor lattice QCD*, *Phys. Rev.* **D99** (2019) 114509 [1809.02827].

[11] [PACS 19] J. Kakazu, K.-i. Ishikawa, N. Ishizuka, Y. Kuramashi, Y. Nakamura, Y. Namekawa et al., *K_{l3} form factors at the physical point on (10.9 fm)^3 volume*, *Phys. Rev. D* **101** (2020) 094504 [1912.13127].

[12] [PACS 22] K.-I. Ishikawa, N. Ishizuka, Y. Kuramashi, Y. Namekawa, Y. Taniguchi, N. Ukita et al., *$K\ell 3$ form factors at the physical point: Toward the continuum limit*, *Phys. Rev. D* **106** (2022) 094501 [2206.08654].

[13] [ETM 21] C. Alexandrou et al., *Ratio of kaon and pion leptonic decay constants with $N_f = 2+1+1$ Wilson-clover twisted-mass fermions*, *Phys. Rev. D* **104** (2021) 074520 [2104.06747].

[14] [CalLat 20] N. Miller et al., *f_k/f_π from Möbius domain-wall fermions solved on gradient-flowed hisq ensembles*, *Phys. Rev. D* **102** (2020) 034507 [2005.04795].

[15] [χQCD 21] J. Liang, A. Alexandru, Y.-J. Bi, T. Draper, K.-F. Liu and Y.-B. Yang, *Detecting flavors of vacuum from the Dirac operator spectrum*, 2102.05380.

[16] C. Wang, Y. Bi, H. Cai, Y. Chen, M. Gong and Z. Liu, *Quark chiral condensate from the overlap quark propagator*, *Chin. Phys. C* **41** (2017) 053102 [1612.04579].

[17] [ETM 20A] M. Fischer, B. Kostrzewa, M. Mai, M. Petschlies, F. Pittler, M. Ueding et al., *The ρ-resonance with physical pion mass from $N_f = 2$ lattice QCD*, *Phys. Lett. B* **819** (2021) 136449 [2006.13805].

[18] X. Gao, N. Karthik, S. Mukherjee, P. Petreczky, S. Syritsyn and Y. Zhao, *Pion form factor and charge radius from Lattice QCD at physical point*, *Phys. Rev. D* **104** (2021) 114515 [2102.06047].

[19] [χQCD 20] G. Wang, J. Liang, T. Draper, K.-F. Liu and Y.-B. Yang, *Lattice Calculation of Pion Form Factor with Overlap Fermions*, *Phys. Rev. D* **104** (2021) 074502 [2006.05431].

[20] X. Feng, Y. Fu and L.-C. Jin, *Lattice QCD calculation of the pion charge radius using a model-independent method*, *Phys. Rev. D* **101** (2020) 051502 [1911.04064].

[21] [ETM 20B] M. Fischer, B. Kostrzewa, L. Liu, F. Romero-López, M. Ueding and C. Urbach, *Scattering of two and three physical pions at maximal isospin from lattice QCD*, *Eur. Phys. J. C* **81** (2021) 436 [2008.03035].

[22] M. Mai, C. Culver, A. Alexandru, M. Döring and F.X. Lee, *Cross-channel study of pion scattering from lattice QCD*, *Phys. Rev. D* **100** (2019) 114514 [1908.01847].

[23] C. Culver, M. Mai, A. Alexandru, M. Döring and F.X. Lee, *Pion scattering in the isospin $I = 2$ channel from elongated lattices*, *Phys. Rev. D* **100** (2019) 034509 [1905.10202].

[24] [ETM 18B] C. Helmes, C. Jost, B. Knippschild, B. Kostrzewa, L. Liu, F. Pittler et al., *Hadron-Hadron Interactions from $N_f = 2+1+1$ Lattice QCD: $I = 3/2$ πK Scattering Length*, *Phys. Rev. D* **98** (2018) 114511 [1809.08886].

[25] [ETM 17G] C. Helmes, C. Jost, B. Knippschild, B. Kostrzewa, L. Liu, C. Urbach et al., *Hadron-Hadron Interactions from $N_f = 2+1+1$ lattice QCD: Isospin-1 KK scattering length*, *Phys. Rev. D* **96** (2017) 034510 [1703.04737].

[26] [PACS-CS 13] K. Sasaki, N. Ishizuka, M. Oka and T. Yamazaki, *Scattering lengths for two pseudoscalar meson systems*, *Phys. Rev.* **D89** (2014) 054502 [1311.7226].

[27] Z. Fu, *Lattice study on πK scattering with moving wall source*, *Phys. Rev. D* **85** (2012) 074501 [1110.1422].

[28] [NPLQCD 07B] S. R. Beane et al., *The K+K+ scattering length from lattice QCD*, *Phys. Rev.* **D77** (2008) 094507 [0709.1169].

[29] [NPLQCD 06B] S. R. Beane, P.F. Bedaque, T.C. Luu, K. Orginos, E. Pallante, A. Parreno et al., *pi K scattering in full QCD with domain-wall valence quarks*, *Phys. Rev. D* **74** (2006) 114503 [hep-lat/0607036].

[30] [RBC/UKQCD 20] R. Abbott et al., *Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay from the Standard Model*, *Phys. Rev. D* **102** (2020) 054509 [2004.09440].

[31] [RBC/UKQCD 15G] Z. Bai et al., *Standard Model Prediction for Direct CP Violation in $K \to \pi\pi$ Decay*, *Phys. Rev. Lett.* **115** (2015) 212001 [1505.07863].

[32] [RBC/UKQCD 15F] T. Blum et al., *$K \to \pi\pi$ $\Delta I = 3/2$ decay amplitude in the continuum limit*, *Phys. Rev.* **D91** (2015) 074502 [1502.00263].

[33] [χQCD 20A] Y. Chen, W.-F. Chiu, M. Gong, Z. Liu and Y. Ma, *Charmed and ϕ meson decay constants from 2+1-flavor lattice QCD*, *Chin. Phys. C* **45** (2021) 023109 [2008.05208].

[34] [RBC/UKQCD 17] P. A. Boyle, L. Del Debbio, A. Jüttner, A. Khamseh, F. Sanfilippo and J.T. Tsang, *The decay constants $\mathbf{f_D}$ and $\mathbf{f_{D_s}}$ in the continuum limit of $\mathbf{N_f = 2+1}$ domain wall lattice QCD*, *JHEP* **12** (2017) 008 [1701.02644].

[35] [RBC/UKQCD 18A] P. A. Boyle, L. Del Debbio, N. Garron, A. Juttner, A. Soni, J.T. Tsang et al., *SU(3)-breaking ratios for $D_{(s)}$ and $B_{(s)}$ mesons*, 1812.08791.

[36] B. Blossier, J. Heitger and M. Post, *Leptonic D_s decays in two-flavour lattice QCD*, *Phys. Rev.* **D98** (2018) 054506 [1803.03065].

[37] R. Balasubramamian and B. Blossier, *Decay constant of B_s and B_s^* mesons from* $\mathrm{N_f} = 2$ *lattice QCD*, *Eur. Phys. J. C* **80** (2020) 412 [1912.09937].

[38] [FNAL/MILC 14A] A. Bazavov et al., *Charmed and light pseudoscalar meson decay constants from four-flavor lattice QCD with physical light quarks*, *Phys.Rev.* **D90** (2014) 074509 [1407.3772].

[39] [χQCD 14] Y. Yi-Bo et al., *Charm and strange quark masses and f_D_s from overlap fermions*, *Phys. Rev.* **D92** (2015) 034517 [1410.3343].

[40] [HPQCD 21A] B. Chakraborty, W.G. Parrott, C. Bouchard, C.T.H. Davies, J. Koponen and G.P. Lepage, *Improved V_{cs} determination using precise lattice QCD form factors for $D \to K\ell\nu$*, *Phys. Rev. D* **104** (2021) 034505 [2104.09883].

[41] [HPQCD 13A] R. Dowdall, C. Davies, G. Lepage and C. McNeile, *V_{us} from π and K decay constants in full lattice QCD with physical u, d, s and c quarks*, *Phys.Rev.* **D88** (2013) 074504 [1303.1670].

[42] [HPQCD 14A] B. Chakraborty, C.T.H. Davies, G.C. Donald, R.J. Dowdall, B. Galloway, P. Knecht et al., *High-precision quark masses and QCD coupling from $n_f = 4$ lattice QCD*, *Phys.Rev.* **D91** (2015) 054508 [1408.4169].

[43] B. Chakraborty, C.T.H. Davies, P.G. de Oliviera, J. Koponen, G.P. Lepage and R.S. Van de Water, *The hadronic vacuum polarization contribution to a_μ from full lattice QCD*, *Phys. Rev. D* **96** (2017) 034516 [1601.03071].

[44] Q.-A. Zhang, J. Hua, F. Huang, R. Li, Y. Li, C.-D. Lu et al., *$\Xi_c \to \Xi$ Form Factors and $\Xi_c \to \Xi\ell^+\nu_\ell$ Decay Rates From Lattice QCD*, *Chin. Phys. C* **46** (2022) 011002 [2103.07064].

[45] [HPQCD 20] L. J. Cooper, C.T.H. Davies, J. Harrison, J. Komijani and M. Wingate, *$B_c \to B_{s(d)}$ form factors from lattice QCD*, *Phys. Rev. D* **102** (2020) 014513 [2003.00914], [Erratum: Phys.Rev.D 103, 099901 (2021)].

[46] [HPQCD 19A] R. J. Dowdall, C.T.H. Davies, R.R. Horgan, G.P. Lepage, C.J. Monahan, J. Shigemitsu et al., *Neutral B-meson mixing from full lattice QCD at the physical point*, *Phys. Rev.* **D100** (2019) 094508 [1907.01025].

[47] [HPQCD 11B] R. J. Dowdall et al., *The upsilon spectrum and the determination of the lattice spacing from lattice QCD including charm quarks in the sea*, *Phys.Rev.* **D85** (2012) 054509 [1110.6887].

[48] [RBC/UKQCD 10A] Y. Aoki et al., *Continuum limit physics from 2+1 flavor domain wall QCD*, *Phys.Rev.* **D83** (2011) 074508 [1011.0892].

[49] [RBC/UKQCD 14B] T. Blum et al., *Domain wall QCD with physical quark masses*, *Phys. Rev.* **D93** (2016) 074505 [1411.7017].

[50] [RBC/UKQCD 23] J. M. Flynn, R.C. Hill, A. Jüttner, A. Soni, J.T. Tsang and O. Witzel, *Exclusive semileptonic $B_s \to K\ell\nu$ decays on the lattice*, *Phys. Rev. D* **107** (2023) 114512 [2303.11280].

[51] [JLQCD 22] B. Colquhoun, S. Hashimoto, T. Kaneko and J. Koponen, *Form factors of $B \to \pi\ell\nu$ and a determination of $|V_{ub}|$ with Möbius domain-wall fermions*, *Phys. Rev. D* **106** (2022) 054502 [2203.04938].

[52] [BMW 12A] S. Borsanyi, S. Dürr, Z. Fodor, C. Hoelbling, S.D. Katz et al., *High-precision scale setting in lattice QCD*, *JHEP* **1209** (2012) 010 [1203.4469].

[53] [FNAL/MILC 19] A. Bazavov et al., *$B_s \to K\ell\nu$ decay from lattice QCD*, *Phys. Rev. D* **100** (2019) 034501 [1901.02561].

[54] [FNAL/MILC 11] A. Bazavov et al., *B- and D-meson decay constants from three-flavor lattice QCD*, *Phys.Rev.* **D85** (2012) 114506 [1112.3051].

[55] S. Meinel and G. Rendon, *$\Lambda_b \to \Lambda^*(1520)\ell^+\ell^-$ form factors from lattice QCD*, *Phys. Rev. D* **103** (2021) 074505 [2009.09313].

[56] S. Meinel and G. Rendon, *$\Lambda_b \to \Lambda_c^*(2595, 2625)\ell^-\bar{\nu}$ form factors from lattice QCD*, *Phys. Rev. D* **103** (2021) 094516 [2103.08775].

[57] [HPQCD 19] E. McLean, C.T.H. Davies, J. Koponen and A.T. Lytle, *$B_s \to D_s\ell\nu$ Form Factors for the full q^2 range from Lattice QCD with non-perturbatively normalized currents*, *Phys. Rev. D* **101** (2020) 074513 [1906.00701].

[58] [HPQCD 19B] E. McLean, C.T.H. Davies, A.T. Lytle and J. Koponen, *Lattice QCD form factor for $B_s \to D_s^* l\nu$ at zero recoil with non-perturbative current renormalisation*, *Phys. Rev. D* **99** (2019) 114512 [1904.02046].

[59] [FNAL/MILC 21] A. Bazavov et al., *Semileptonic form factors for $B \to D^*\ell\nu$ at nonzero recoil from $2 + 1$-flavor lattice QCD: Fermilab Lattice and MILC Collaborations*, *Eur. Phys. J. C* **82** (2022) 1141 [2105.14019], [Erratum: Eur.Phys.J.C 83, 21 (2023)].

[60] [FNAL/MILC 17] A. Bazavov et al., *B- and D-meson leptonic decay constants from four-flavor lattice QCD*, *Phys. Rev.* **D98** (2018) 074512 [1712.09262].

[61] A. Nada and A. Ramos, *An analysis of systematic effects in finite size scaling studies using the gradient flow*, *Eur. Phys. J. C* **81** (2021) 1 [2007.12862].

[62] M. Dalla Brida and A. Ramos, *The gradient flow coupling at high-energy and the scale of SU(3) Yang–Mills theory*, *Eur. Phys. J. C* **79** (2019) 720 [1905.05147].

[63] N. Husung, A. Nada and R. Sommer, *Yang Mills short distance potential and perturbation theory*, *PoS* **LATTICE2019** (2020) 263.

[64] P. Petreczky and J.H. Weber, *Strong coupling constant from moments of quarkonium correlators revisited*, *Eur. Phys. J. C* **82** (2022) 64 [2012.06193].

[65] C. Ayala, X. Lobregat and A. Pineda, *Determination of $\alpha(M_z)$ from an hyperasymptotic approximation to the energy of a static quark-antiquark pair*, *JHEP* **09** (2020) 016 [2005.12301].

[66] S. Cali, K. Cichy, P. Korcyl and J. Simeth, *Running coupling constant from position-space current-current correlation functions in three-flavor lattice QCD*, *Phys. Rev. Lett.* **125** (2020) 242002 [2003.05781].

[67] D. Boito and V. Mateu, *Precise determination of α_s from relativistic quarkonium sum rules*, *JHEP* **03** (2020) 094 [2001.11041].

[68] [TUMQCD 19] A. Bazavov, N. Brambilla, X. Garcia i Tormo, P. Petreczky, J. Soto, A. Vairo et al., *Determination of the QCD coupling from the static energy and the free energy*, *Phys. Rev. D* **100** (2019) 114511 [1907.11747].

[69] S. Zafeiropoulos, P. Boucaud, F. De Soto, J. Rodríguez-Quintero and J. Segovia, *Strong Running Coupling from the Gauge Sector of Domain Wall Lattice QCD with Physical Quark Masses*, *Phys. Rev. Lett.* **122** (2019) 162002 [1902.08148].

[70] [CalLat 19] A. Walker-Loud et al., *Lattice QCD Determination of g_A*, *PoS* **CD2018** (2020) 020 [1912.08321].

[71] [ETM 19] C. Alexandrou, S. Bacchio, M. Constantinou, J. Finkenrath, K. Hadjiyiannakou, K. Jansen et al., *Nucleon axial, tensor, and scalar charges and σ-terms in lattice QCD*, *Phys. Rev. D* **102** (2020) 054517 [1909.00485].

[72] [χQCD 21A] L. Liu, T. Chen, T. Draper, J. Liang, K.-F. Liu, G. Wang et al., *Nucleon isovector scalar charge from overlap fermions*, *Phys. Rev. D* **104** (2021) 094503 [2103.12933].

[73] [NME 21] S. Park, R. Gupta, B. Yoon, S. Mondal, T. Bhattacharya, Y.-C. Jang et al., *Precision Nucleon Charges and Form Factors Using 2+1-flavor Lattice QCD*, *Phys. Rev. D* **105** (2022) 054505 [2103.05599].

[74] [RBC/UKQCD 19] M. Abramczyk, T. Blum, T. Izubuchi, C. Jung, M. Lin, A. Lytle et al., *Nucleon mass and isovector couplings in 2+1-flavor dynamical domain-wall lattice QCD near physical mass*, *Phys. Rev. D* **101** (2020) 034510 [1911.03524].

[75] [LHPC 19] N. Hasan, J. Green, S. Meinel, M. Engelhardt, S. Krieg, J. Negele et al., *Nucleon axial, scalar, and tensor charges using lattice QCD at the physical pion mass*, *Phys. Rev. D* **99** (2019) 114505 [1903.06487].

[76] [Mainz 19] T. Harris, G. von Hippel, P. Junnarkar, H.B. Meyer, K. Ottnad, J. Wilhelm et al., *Nucleon isovector charges and twist-2 matrix elements with $N_f = 2+1$ dynamical Wilson quarks*, *Phys. Rev. D* **100** (2019) 034513 [1905.01291].

[77] [PACS 18A] E. Shintani, K.-I. Ishikawa, Y. Kuramashi, S. Sasaki and T. Yamazaki, *Nucleon form factors and root-mean-square radii on a (10.8 fm)^4 lattice at the physical point*, *Phys. Rev. D* **99** (2019) 014510 [1811.07292], [Erratum: Phys.Rev.D 102, 019902 (2020)].

[78] [BMW 20A] Sz. Borsanyi, Z. Fodor, C. Hoelbling, L. Lellouch, K. Szabo, C. Torrero et al., *Ab-initio calculation of the proton and the neutron's scalar couplings for new physics searches*, 2007.03319.

[79] A. Walker-Loud et al., *Light hadron spectroscopy using domain wall valence quarks on an Asqtad sea*, *Phys. Rev. D* **79** (2009) 054502 [0806.4549].

[80] B.C. Tiburzi and A. Walker-Loud, *Strong isospin breaking in the nucleon and Delta masses*, *Nucl. Phys. A* **764** (2006) 274 [hep-lat/0501018].

[81] S.R. Beane, *Nucleon masses and magnetic moments in a finite volume*, *Phys. Rev.* **D70** (2004) 034507 [hep-lat/0403015].

[82] [CalLat 20A] N. Miller et al., *Scale setting the Möbius domain wall fermion on gradient-flowed HISQ action using the omega baryon mass and the gradient-flow scales t_0 and w_0*, *Phys. Rev. D* **103** (2021) 054511 [2011.12166].

[83] [BMW 20] Sz. Borsanyi et al., *Leading hadronic contribution to the muon magnetic moment from lattice QCD*, *Nature* **593** (2021) 51 [2002.12347].

[84] [ETM 20] G. Bergner, P. Dimopoulos, J. Finkenrath, E. Fiorenza, R. Frezzotti, M. Garofalo et al., *Quark masses and decay constants in $N_f = 2+1+1$ isoQCD with Wilson clover twisted mass fermions*, in *37th International Symposium on Lattice Field Theory (Lattice 2019) Wuhan, Hubei, China, June 16-22, 2019*, vol. LATTICE2019, p. 181, 2020, DOI [2001.09116].

[85] [ETM 18A] C. Alexandrou et al., *Simulating twisted mass fermions at physical light, strange and charm quark masses*, *Phys. Rev.* **D98** (2018) 054518 [1807.00495].

[86] [MILC 15] A. Bazavov et al., *Gradient flow and scale setting on MILC HISQ ensembles*, *Phys. Rev.* **D93** (2016) 094510 [1503.02769].

[87] [ETM 14] N. Carrasco et al., *Up, down, strange and charm quark masses with N_f = 2+1+1 twisted mass lattice QCD*, *Nucl. Phys.* **B887** (2014) 19 [1403.4504].

[88] [RQCD 22] G. S. Bali, S. Collins, P. Georg, D. Jenkins, P. Korcyl, A. Schäfer et al., *Scale setting and the light baryon spectrum in $N_f = 2+1$ QCD with Wilson fermions*, 2211.03744.

[89] [CLS 21] B. Strassberger et al., *Scale setting for CLS 2+1 simulations*, *PoS* **LATTICE2021** (2022) 135 [2112.06696].

[90] [CLS 16] M. Bruno, T. Korzec and S. Schaefer, *Setting the scale for the CLS 2+1 flavor ensembles*, *Phys. Rev.* **D95** (2017) 074504 [1608.08900].

[91] [QCDSF/UKQCD 15B] V. .G. Bornyakov et al., *Wilson flow and scale setting from lattice QCD*, 1508.05916.

[92] [HotQCD 14] A. Bazavov et al., *Equation of state in (2+1)-flavor QCD*, *Phys.Rev.* **D90** (2014) 094503 [1407.6387].

[93] [HotQCD 11] A. Bazavov, T. Bhattacharya, M. Cheng, C. DeTar, H. Ding et al., *The chiral and deconfinement aspects of the QCD transition*, *Phys.Rev.* **D85** (2012) 054503 [1111.1710].

[94] [MILC 10] A. Bazavov et al., *Results for light pseudoscalar mesons*, *PoS* **LAT2010** (2010) 074 [1012.0868].

[95] [HPQCD 09B] C. T. H. Davies, E. Follana, I. Kendall, G.P. Lepage and C. McNeile, *Precise determination of the lattice spacing in full lattice QCD*, *Phys.Rev.* **D81** (2010) 034506 [0910.1229].

[96] [MILC 09] A. Bazavov et al., *Full nonperturbative QCD simulations with 2+1 flavors of improved staggered quarks*, *Rev. Mod. Phys.* **82** (2010) 1349 [0903.3598].

[97] [MILC 09A] A. Bazavov et al., *MILC results for light pseudoscalars*, *PoS* **CD09** (2009) 007 [0910.2966].

[98] [MILC 09B] A. Bazavov et al., *Results from the MILC collaboration's SU(3) chiral perturbation theory analysis*, *PoS* **LAT2009** (2009) 079 [0910.3618].

[99] [PACS-CS 08] S. Aoki et al., *2+1 flavor lattice QCD toward the physical point*, *Phys. Rev.* **D79** (2009) 034503 [0807.1661].

[100] [HPQCD 05B] A. Gray et al., *The upsilon spectrum and m_b from full lattice QCD*, *Phys.Rev.* **D72** (2005) 094507 [hep-lat/0507013].

[101] C. Aubin et al., *Light hadrons with improved staggered quarks: Approaching the continuum limit*, *Phys. Rev.* **D70** (2004) 094505 [hep-lat/0402030].